Sum of squared logarithms - An inequality relating positive definite matrices and their matrix logarithm

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Abstract

Let $y_1, y_2, y_3, a_1, a_2, a_3 \in (0, \infty)$ be such that $y_1\, y_2\, y_3 = a_1\, a_2\, a_3$ and

$$y_1 + y_2 + y_3 \geq a_1 + a_2 + a_3, \quad y_1 y_2 + y_2 y_3 + y_1 y_3 \geq a_1 a_2 + a_2 a_3 + a_1 a_3.$$

Then

$$(\log y_1)^2 + (\log y_2)^2 + (\log y_3)^2 \geq (\log a_1)^2 + (\log a_2)^2 + (\log a_3)^2.$$

This can also be stated in terms of real positive definite 3×3-matrices P_1, P_2: If their determinants are equal $\det P_1 = \det P_2$, then

$$\operatorname{tr} P_1 \geq \operatorname{tr} P_2 \text{ and } \operatorname{tr} \operatorname{Cof} P_1 \geq \operatorname{tr} \operatorname{Cof} P_2 \implies \|\log P_1\|_F^2 \geq \|\log P_2\|_F^2\,,$$

where log is the principal matrix logarithm and $\|P\|_F^2 = \sum_{i,j=1}^3 P_{ij}^2$ denotes the Frobenius matrix norm. Applications in matrix analysis and nonlinear elasticity are indicated.

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1 Introduction

Convexity is a powerful source for obtaining new inequalities, see e.g. [1, 8]. In applications coming from nonlinear elasticity we are faced, however, with variants of the squared logarithm function, see the last section. The function $(\log(x))^2$ is neither convex nor concave. Nevertheless the sum of squared logarithm inequality holds. We will proceed as follows: In the first section we will give several equivalent formulations of the inequality, for example in terms of the coefficients of the characteristic polynomial (Theorem 1), in terms of elementary symmetric polynomials (Theorem 3), in terms of means (Theorem 5) or in terms of the Frobenius matrix norm (Theorem 7). A proof of the inequality will be given in Section 2, and some counterexamples for slightly changed variants of the inequality are discussed in Section 3. In the last section an application of the sum of squared logarithms inequality in matrix analysis and in the mathematical theory of nonlinear elasticity is indicated.

2 Formulations of the problem

All theorems in this section are equivalent.

Theorem 1. *For $n = 2$ or $n = 3$ let $P_1, P_2 \in \mathbb{R}^{n\times n}$ be positive definite real matrices. Let the coefficients of the characteristic polynomials of P_1 and P_2 satisfy*

$$\operatorname{tr} P_1 \geq \operatorname{tr} P_2 \ \textit{and} \ \operatorname{tr}\operatorname{Cof} P_1 \geq \operatorname{tr}\operatorname{Cof} P_2, \ \textit{and} \ \det P_1 = \det P_2.$$

Then

$$\|\log P_1\|_F^2 \geq \|\log P_2\|_F^2 \,.$$

For $n = 3$ we will now give equivalent formulations of this statement. The case $n = 2$ can be treated analogously. For its proof see Remark 15. By orthogonal diagonalization of P_1 and P_2 the inequalities can be rewritten in terms of the eigenvalues y_1, y_2, y_3 and a_1, a_2, a_3 respectively.

Theorem 2. *Let the real numbers $a_1, a_2, a_3 > 0$ and $y_1, y_2, y_3 > 0$ be such that*

$$\begin{aligned} y_1 + y_2 + y_3 &\geq a_1 + a_2 + a_3, \\ y_1y_2 + y_2y_3 + y_1y_3 &\geq a_1a_2 + a_2a_3 + a_1a_3, \\ y_1y_2y_3 &= a_1a_2a_3. \end{aligned} \tag{1}$$

Then

$$(\log y_1)^2 + (\log y_2)^2 + (\log y_3)^2 \geq (\log a_1)^2 + (\log a_2)^2 + (\log a_3)^2. \tag{2}$$

The elementary symmetric polynomials, see e.g. [9, p.178]

$$\begin{aligned}
e_0(y_1,y_2,y_3) &= 1\\
e_1(y_1,y_2,y_3) &= y_1+y_2+y_3\\
e_2(y_1,y_2,y_3) &= y_1y_2+y_1y_3+y_2y_3\\
e_3(y_1,y_2,y_3) &= y_1y_2y_3
\end{aligned}$$

are known to have the Schur-concavity property (i.e. $-e_k$ is Schur-convex) [1, 5], see (16). It is possible to express the problem in terms of these elementary symmetric polynomials as follows:

Theorem 3. *Let* $a_1,a_2,a_3>0$ *and* $y_1,y_2,y_3>0$ *satisfy*

$$e_1(y_1,y_2,y_3)\ge e_1(a_1,a_2,a_3),\quad e_2(y_1,y_2,y_3)\ge e_2(a_1,a_2,a_3),\quad e_3(y_1.y_2,y_3)=e_3(a_1,a_2,a_3).$$

Then

$$e_1((\log y_1)^2,(\log y_2)^2,(\log y_3)^2)\ge e_1((\log a_1)^2,(\log a_2)^2,(\log a_3)^2).$$

Because $y_1y_2y_3=a_1a_2a_3>0$, we have

$$y_1y_2+y_2y_3+y_1y_3\ge a_1a_2+a_2a_3+a_1a_3 \quad\Leftrightarrow\quad \frac{1}{y_1}+\frac{1}{y_2}+\frac{1}{y_3}\ge\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}.$$

Thus we obtain

Theorem 4. *Let the real numbers* $a_1,a_2,a_3>0$ *and* $y_1,y_2,y_3>0$ *be such that*

$$\begin{gathered}
y_1+y_2+y_3\ge a_1+a_2+a_3,\\
\frac{1}{y_1}+\frac{1}{y_2}+\frac{1}{y_3}\ge\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3},\\
y_1y_2y_3=a_1a_2a_3.
\end{gathered}\tag{3}$$

Then

$$(\log y_1)^2+(\log y_2)^2+(\log y_3)^2\ge(\log a_1)^2+(\log a_2)^2+(\log a_3)^2.\tag{4}$$

The conditions (3) are also simple expressions in terms of arithmetic, harmonic and geometric and quadratic mean

$$\begin{gathered}
A(y_1,y_2,y_3)=\frac{y_1+y_2+y_3}{3},\quad H(y_1,y_2,y_3)=\frac{3}{\frac{1}{y_1}+\frac{1}{y_2}+\frac{1}{y_3}},\\
G(y_1,y_2,y_3)=\sqrt[3]{y_1y_2y_3},\quad Q(y_1,y_2,y_3)=\sqrt{\frac{1}{3}(y_1^2+y_2^2+y_3^2)}
\end{gathered}$$

Theorem 5. *Let $a_1, a_2, a_3 > 0$ and $y_1, y_2, y_3 > 0$. Then $A(y_1, y_2, y_3) \geq A(a_1, a_2, a_3)$, $H(a_1, a_2, a_3) \geq H(y_1, y_2, y_3)$ ("reverse!") and $G(y_1, y_2, y_3) = G(a_1, a_2, a_3)$ imply*

$$Q(\log y_1, \log y_2, \log y_3) \geq Q(\log a_1, \log a_2, \log a_3)$$

We denote by

$$a_i =: d_i^2\,, \qquad y_i =: x_i^2.$$

and arrive at

Theorem 6. *Let the real numbers d_i and x_i be such that $d_1, d_2, d_3 > 0$, $x_1, x_2, x_3 > 0$ and*

$$\begin{aligned} &x_1^2 + x_2^2 + x_3^2 \geq d_1^2 + d_2^2 + d_3^2, \\ &x_1^2 x_2^2 + x_2^2 x_3^2 + x_1^2 x_3^2 \geq d_1^2 d_2^2 + d_2^2 d_3^2 + d_1^2 d_3^2, \\ &x_1 x_2 x_3 \;=\; d_1 d_2 d_3. \end{aligned} \tag{5}$$

Then

$$(\log x_1)^2 + (\log x_2)^2 + (\log x_3)^2 \geq (\log d_1)^2 + (\log d_2)^2 + (\log d_3)^2. \tag{6}$$

If we again view x_i and d_i as eigenvalues of positive definite matrices, an equivalent formulation of the problem can be given in terms of their Frobenius matrix norms:

Theorem 7. *For $n \in \{2, 3\}$ let $P_1, P_2 \in \mathbb{R}^{n\times n}$ be positive definite real matrices. Let*

$$\|P_1\|_F^2 \geq \|P_2\|_F^2 \;\; \textit{and} \;\; \left\|P_1^{-1}\right\|_F^2 \geq \left\|P_2^{-1}\right\|_F^2, \;\; \textit{and} \;\; \det P_1 = \det P_2.$$

Then

$$\|\log P_1\|_F^2 \geq \|\log P_2\|_F^2\,.$$

Let us reconsider the formulation from Theorem 5. If we denote

$$c_i := \log a_i\,, \qquad z_i := \log y_i\,,$$

from $H(a_1, a_2, a_3) \geq H(y_1, y_2, y_3)$ we obtain

$$e^{-z_1} + e^{-z_2} + e^{-z_3} \geq e^{-c_1} + e^{-c_2} + e^{-c_3}.$$

Theorem 8. *Let the real numbers c_1, c_2, c_3 and z_1, z_2, z_3 be such that*

$$\begin{aligned} &e^{z_1} + e^{z_2} + e^{z_3} \geq e^{c_1} + e^{c_2} + e^{c_3}, \\ &e^{-z_1} + e^{-z_2} + e^{-z_3} \geq e^{-c_1} + e^{-c_2} + e^{-c_3}, \\ &z_1 + z_2 + z_3 = c_1 + c_2 + c_3. \end{aligned} \tag{7}$$

Then

$$z_1^2 + z_2^2 + z_3^2 \geq c_1^2 + c_2^2 + c_3^2\,. \tag{8}$$

In order to prove Theorem 8, one can assume without loss of generality that

$$z_1 + z_2 + z_3 = c_1 + c_2 + c_3 = 0. \tag{9}$$

Thus, we have the equivalent formulation

Theorem 9. *Let the real numbers* $\bar{c}_1, \bar{c}_2, \bar{c}_3$ *and* $\bar{z}_1, \bar{z}_2, \bar{z}_3$ *be such that*

$$\begin{aligned} &e^{\bar{z}_1} + e^{\bar{z}_2} + e^{\bar{z}_3} \geq e^{\bar{c}_1} + e^{\bar{c}_2} + e^{\bar{c}_3}, \\ &e^{-\bar{z}_1} + e^{-\bar{z}_2} + e^{-\bar{z}_3} \geq e^{-\bar{c}_1} + e^{-\bar{c}_2} + e^{-\bar{c}_3}, \\ &\bar{z}_1 + \bar{z}_2 + \bar{z}_3 = \bar{c}_1 + \bar{c}_2 + \bar{c}_3 = 0. \end{aligned} \tag{10}$$

Then

$$\bar{z}_1^2 + \bar{z}_2^2 + \bar{z}_3^2 \geq \bar{c}_1^2 + \bar{c}_2^2 + \bar{c}_3^2 \,. \tag{11}$$

Let us prove that Theorem 8 can be reformulated as Theorem 9. Indeed, let us assume that Theorem 9 is valid and show that the statement of Theorem 8 also holds true. We denote by s the sum $s = z_1 + z_2 + z_3 = c_1 + c_2 + c_3$ and we designate

$$\bar{z}_i = z_i - \frac{s}{3}, \qquad \bar{c}_i = c_i - \frac{s}{3} \qquad (i = 1, 2, 3).$$

Then, the real numbers $\bar{z}_i$ and $\bar{c}_i$ satisfy the hypotheses of Theorem 9 and we obtain $\bar{z}_1^2 + \bar{z}_2^2 + \bar{z}_3^2 \geq \bar{c}_1^2 + \bar{c}_2^2 + \bar{c}_3^2$. This inequality is equivalent to

$$\sum_{i=1}^{3} \left(z_i - \frac{s}{3}\right)^2 \geq \sum_{i=1}^{3} \left(c_i - \frac{s}{3}\right)^2,$$

which, by virtue of the condition $(7)_3$, reduces to

$$z_1^2 + z_2^2 + z_3^2 \geq c_1^2 + c_2^2 + c_3^2 \,.$$

Thus, Theorem 8 is also valid.

By virtue of the logical equivalence

$$(A \wedge B \;\Rightarrow\; C) \qquad \Leftrightarrow \qquad (\neg C \;\Rightarrow\; \neg A \vee \neg B)$$

for any statements A, B, C, we can formulate the inequality (11) (i.e., Theorem 9) in the following equivalent manner:

Theorem 10. *Let the real numbers* c_1, c_2, c_3 *and* z_1, z_2, z_3 *be such that*

$$z_1 + z_2 + z_3 = c_1 + c_2 + c_3 = 0 \qquad \text{and} \qquad z_1^2 + z_2^2 + z_3^2 < c_1^2 + c_2^2 + c_3^2 \,. \tag{12}$$

Then one of the following inequalities holds:

$$\begin{aligned} &e^{z_1} + e^{z_2} + e^{z_3} < e^{c_1} + e^{c_2} + e^{c_3} \qquad \text{or} \\ &e^{-z_1} + e^{-z_2} + e^{-z_3} < e^{-c_1} + e^{-c_2} + e^{-c_3}. \end{aligned} \tag{13}$$

We use the statement of Theorem 10 for the proof.

Before continuing let us show that our new inequality is not a consequence of majorization and Karamata's inequality [4]. Consider $z = (z_1, \ldots, z_n) \in \mathbb{R}^n_+$ and $c = (c_1, \ldots, c_n) \in \mathbb{R}^n_+$ arranged already in decreasing order $z_1 \geq z_2 \geq \ldots \geq z_n$ and $c_1 \geq c_2 \geq \ldots \geq c_n$. If

$$\sum_{i=1}^{k} z_i \geq \sum_{i=1}^{k} c_i\,, \quad (1 \leq k \leq n-1)\,, \quad \sum_{i=1}^{n} z_i = \sum_{i=1}^{n} c_i\,, \tag{14}$$

we say that z majorizes c, denoted by $z \succ c$. The following result is well known [4, 5][2, p.89]. If $f : \mathbb{R} \mapsto \mathbb{R}$ is convex, then

$$z \succ c \quad \Rightarrow \quad \sum_{i=1}^{n} f(z_i) \geq \sum_{i=1}^{n} f(c_i)\,. \tag{15}$$

A function $g : \mathbb{R}^n \mapsto \mathbb{R}$ which satisfies

$$z \succ c \quad \Rightarrow \quad g(z_1, \ldots, z_n) \geq g(c_1, \ldots, c_n) \tag{16}$$

is called Schur-convex. In Theorem 8 the convex function to be considered would be $f(t) = t^2$. Do conditions (7) (upon rearrangement of $z, c \in \mathbb{R}^3_+$ if necessary) yield already majorization $z \succ c$? This is not the case, as we explain now. Let the real numbers $z_1 \geq z_2 \geq z_3$ and $c_1 \geq c_2 \geq c_3$ be such that

$$\begin{aligned} &e^{z_1} + e^{z_2} + e^{z_3} \geq e^{c_1} + e^{c_2} + e^{c_3}, \\ &e^{-z_1} + e^{-z_2} + e^{-z_3} \geq e^{-c_1} + e^{-c_2} + e^{-c_3}, \\ &z_1 + z_2 + z_3 = c_1 + c_2 + c_3\,. \end{aligned} \tag{17}$$

These conditions do not imply the majorization $z \succ c$,

$$z_1 \geq c_1\,, \qquad z_1 + z_2 \geq c_1 + c_2\,, \qquad z_1 + z_2 + z_3 = c_1 + c_2 + c_3\,. \tag{18}$$

Therefore, our inequality (i.e. $z_1^2 + z_2^2 + z_3^2 \geq c_1^2 + c_2^2 + c_3^2$) does not follow from majorization in disguise.

Indeed, let

$$z_1 = \frac{1}{2} + \frac{0.95}{2\sqrt{3}}\,, \qquad z_2 = \frac{1}{2} + \frac{0.85}{2\sqrt{3}}\,, \qquad z_3 = -1 - \frac{0.9}{\sqrt{3}}\,.$$

and

$$c_1 = \frac{1}{2} + \frac{1}{2\sqrt{3}}\,, \qquad c_2 = -\frac{1}{2} + \frac{1}{2\sqrt{3}}\,, \qquad c_3 = -\frac{1}{\sqrt{3}}\,,$$

Then, we have $z_1 > z_2 > z_3$ and $c_1 > c_2 > c_3$, together with

$$\begin{aligned} &e^{z_1} + e^{z_2} + e^{z_3} = 4.49497... > 3.57137... = e^{c_1} + e^{c_2} + e^{c_3}, \\ &e^{-z_1} + e^{-z_2} + e^{-z_3} = 5.50607... > 3.47107... = e^{-c_1} + e^{-c_2} + e^{-c_3}, \\ &z_1 + z_2 + z_3 = c_1 + c_2 + c_3 = 0, \end{aligned}$$

but the majorization inequalities (18) are not satisfied, since $z_1 < c_1$.

3 Proof of the inequality

Of course we may assume without loss of generality that $c_1 \geq c_2 \geq c_3$ and $z_1 \geq z_2 \geq z_3$ (and the same for a_i, d_i, x_i, y_i).
The proof begins with the crucial

Lemma 11. *Let the real numbers $a \geq b \geq c$ and $x \geq y \geq z$ be such that*

$$a+b+c = x+y+z = 0, \qquad a^2+b^2+c^2 = x^2+y^2+z^2. \tag{19}$$

Then, the inequality

$$e^a + e^b + e^c \leq e^x + e^y + e^z \tag{20}$$

is satisfied if and only if the relation

$$a \leq x \tag{21}$$

holds, or equivalently, if and only if

$$c \leq z \tag{22}$$

holds.

Proof. Let us denote by $r := \sqrt{\frac{2}{3}\,(a^2+b^2+c^2)} > 0$. Then, from (19) it follows

$$\begin{aligned} b+c &= -a, \qquad & b^2+c^2 &= \tfrac{3}{2}\,r^2 - a^2, \\ y+z &= -x, \qquad & y^2+z^2 &= \tfrac{3}{2}\,r^2 - x^2, \end{aligned}$$

and we find

$$\begin{aligned} b &= \tfrac{1}{2}\big(-a+\sqrt{3(r^2-a^2)}\,\big)\,, \qquad & c &= \tfrac{1}{2}\big(-a-\sqrt{3(r^2-a^2)}\,\big)\,, \\ y &= \tfrac{1}{2}\big(-x+\sqrt{3(r^2-x^2)}\,\big)\,, \qquad & z &= \tfrac{1}{2}\big(-x-\sqrt{3(r^2-x^2)}\,\big)\,. \end{aligned} \tag{23}$$

In view of (19) and $a \geq b \geq c$, $x \geq y \geq z$, one can show that

$$a, x \in \big[\frac{r}{2}\,, r\big], \qquad b, y \in \big[-\frac{r}{2}\,, \frac{r}{2}\big], \qquad c, z \in \big[-r\,, -\frac{r}{2}\big]. \tag{24}$$

Indeed, let us verify the relations (24). We have

$$\begin{aligned} &\frac{r}{2} \leq a \leq r \quad \Leftrightarrow \quad \frac{1}{6}\,(a^2+b^2+c^2) \leq a^2 \leq \frac{2}{3}\,(a^2+b^2+c^2) \\ &\Leftrightarrow \quad b^2+c^2 \leq 5a^2 \text{ and } a^2 \leq 2(b^2+c^2) \quad \Leftrightarrow \quad b^2+(a+b)^2 \leq 5a^2 \text{ and } (b+c)^2 \leq 2(b^2+c^2) \\ &\Leftrightarrow \quad 4a^2-2ab-2b^2 \geq 0 \text{ and } b^2+c^2 \geq 2bc \quad \Leftrightarrow \quad 2(a-b)(2a+b) \geq 0 \text{ and } (b-c)^2 \geq 0, \end{aligned}$$

which hold true since $a \geq b$ and $2a + b \geq a + b + c = 0$. Similarly, we have

$$-\frac{r}{2} \leq b \leq \frac{r}{2} \quad \Leftrightarrow \quad b^2 \leq \frac{r^2}{4} \quad \Leftrightarrow \quad 4b^2 \leq \frac{2}{3}(a^2+b^2+c^2) \quad \Leftrightarrow \quad 5b^2 \leq a^2 + c^2$$
$$\Leftrightarrow \quad 5b^2 \leq a^2 + (a+b)^2 \quad \Leftrightarrow \quad 2a^2 + 2ab - 4b^2 \geq 0 \quad \Leftrightarrow \quad 2(a-b)(a+2b) \geq 0,$$

which holds true since $a \geq b$ and $a + 2b \geq a + b + c = 0$. Also, we have

$$-r \leq c \leq -\frac{r}{2} \quad \Leftrightarrow \quad r^2 \geq c^2 \geq \frac{r^2}{4} \quad \Leftrightarrow \quad \frac{2}{3}(a^2+b^2+c^2) \geq c^2 \geq \frac{1}{6}(a^2+b^2+c^2)$$
$$\Leftrightarrow \quad 2(a^2+b^2) \geq c^2 \text{ and } 5c^2 \geq a^2+b^2 \quad \Leftrightarrow \quad 2(a^2+b^2) \geq (a+b)^2 \text{ and } 5(a+b)^2 \geq a^2+b^2$$
$$\Leftrightarrow \quad (a-b)^2 \geq 0 \text{ and } 4a^2+10ab+4b^2 \geq 0 \quad \Leftrightarrow \quad (a-b)^2 \geq 0 \text{ and } 2(a+2b)(2a+b) \geq 0,$$

which hold true since $a+2b \geq a+b+c = 0$ and $2a+b \geq a+b+c = 0$. One can show in the same way that $x \in \big[\frac{r}{2}, r\big]$, $y \in \big[-\frac{r}{2}, \frac{r}{2}\big]$, $z \in \big[-r, -\frac{r}{2}\big]$, so that (24) has been verified.

We prove now that the inequality (21) holds if and only if (22) holds. Indeed, using $(23)_{2,4}$ and (24) we get

$$c \leq z \quad \Leftrightarrow \quad -a - \sqrt{3(r^2-a^2)} \leq -x - \sqrt{3(r^2-x^2)} \quad \Leftrightarrow$$
$$\Leftrightarrow \quad \frac{a}{r} + \sqrt{3\Big(1-\big(\frac{a}{r}\big)^2\Big)} \geq \frac{x}{r} + \sqrt{3\Big(1-\big(\frac{x}{r}\big)^2\Big)} \quad \Leftrightarrow \quad a \leq x\,,$$

since the function $t \mapsto t + \sqrt{3(1-t^2)}$ is decreasing for $t \in \big[\frac{1}{2}, 1\big]$.

Let us prove next that the inequalities (20) and (21) are equivalent. To accomplish this, we introduce the function $f : \big[\frac{r}{2}, r\big] \to \mathbb{R}$ by

$$f(x) = e^x + e^{\left(-x+\sqrt{3(r^2-x^2)}\right)/2} + e^{\left(-x-\sqrt{3(r^2-x^2)}\right)/2}. \tag{25}$$

Taking into account (23) and $(24)_1$, the inequality (20) can be written equivalently as

$$f(a) \leq f(x)\,, \tag{26}$$

which is equivalent to

$$a \leq x\,,$$

since the function f defined by (25) is monotone increasing on $\big[\frac{r}{2}, r\big]$, as we show next. To this aim, we denote by

$$\cos\varphi := \frac{x}{r} \in \big[\frac{1}{2}, 1\big], \qquad \text{i.e.} \qquad \varphi := \arccos\Big(\frac{x}{r}\Big) \in \big[0, \frac{\pi}{3}\big].$$

Then, the function (25) can be written as

$$
\begin{aligned}
&f(x) = h(r,\varphi), \qquad \text{where} \qquad h:(0,\infty)\times\big[\,0,\,\frac{\pi}{3}\,\big]\to\mathbb{R},\\
&h(r,\varphi) = e^{r\cos\varphi} + e^{r\cos(\varphi+2\pi/3)} + e^{r\cos(\varphi-2\pi/3)}.
\end{aligned}
\tag{27}
$$

We have to show that $h(r,\varphi)$ is decreasing with respect to $\varphi\in\big[\,0,\,\frac{\pi}{3}\,\big]$. We compute the first derivative

$$
\frac{\partial}{\partial\varphi}\,h(r,\varphi) = -r\Big[e^{r\cos\varphi}\sin\varphi + e^{r\cos(\varphi+2\pi/3)}\sin(\varphi+\tfrac{2\pi}{3}\,) + e^{r\cos(\varphi-2\pi/3)}\sin(\varphi-\tfrac{2\pi}{3}\,)\Big].
\tag{28}
$$

The function (28) has the same sign as the function

$$
F(r,\varphi) := \frac{1}{r}\,e^{-r\cos\varphi}\,\frac{\partial}{\partial\varphi}\,h(r,\varphi),
\tag{29}
$$

i.e. the function $F:(0,\infty)\times\big[\,0,\,\frac{\pi}{3}\,\big]\to\mathbb{R}$ given by

$$
F(r,\varphi) = -\sin\varphi - e^{-r\sqrt{3}\sin(\varphi+\pi/3)}\sin(\varphi+\tfrac{2\pi}{3}\,) - e^{r\sqrt{3}\sin(\varphi-\pi/3)}\sin(\varphi-\tfrac{2\pi}{3}\,).
\tag{30}
$$

In order to show that $F(r,\varphi)\le 0$ for all $(r,\varphi)\in(0,\infty)\times\big[\,0,\,\frac{\pi}{3}\,\big]$, we remark that $\lim\limits_{r\searrow 0}F(r,\varphi)=0$ for fixed $\varphi\in\big[\,0,\,\frac{\pi}{3}\,\big]$ and we compute

$$
\begin{aligned}
\frac{\partial}{\partial r}\,F(r,\varphi) &= \sqrt{3}\Big[e^{-r\sqrt{3}\sin(\varphi+\pi/3)}\sin(\varphi+\tfrac{\pi}{3})\sin(\varphi+\tfrac{2\pi}{3}) - e^{r\sqrt{3}\sin(\varphi-\pi/3)}\sin(\varphi-\tfrac{\pi}{3})\sin(\varphi-\tfrac{2\pi}{3})\Big]\\
&= \sqrt{3}\Big[e^{-r\sqrt{3}\sin(\varphi+\pi/3)}\frac{1}{2}\big(-\cos(2\varphi+\pi)+\cos\tfrac{\pi}{3}\,\big) - e^{r\sqrt{3}\sin(\varphi-\pi/3)}\frac{1}{2}\big(-\cos(2\varphi-\pi)+\cos\tfrac{-\pi}{3}\,\big)\Big]\\
&= \frac{\sqrt{3}}{2}\,\big(\cos 2\varphi+\frac{1}{2}\big)\Big[e^{-r\sqrt{3}\sin(\varphi+\pi/3)} - e^{r\sqrt{3}\sin(\varphi-\pi/3)}\Big]\ \le 0,
\end{aligned}
$$

since $\varphi\in\big[\,0,\,\frac{\pi}{3}\,\big]$ implies $\cos 2\varphi\ge -\frac{1}{2}$ and $-\sin(\varphi+\frac{\pi}{3}\,)\le\sin(\varphi-\frac{\pi}{3}\,)$.
Consequently, the function $F(r,\varphi)$ is decreasing with respect to r and for any $(r,\varphi)\in(0,\infty)\times\big[\,0,\,\frac{\pi}{3}\,\big]$ we have that

$$
F(r,\varphi)\ \le\ \lim_{r\searrow 0}F(r,\varphi)\ =\ 0\,.
\tag{31}
$$

From (29) and (31) it follows that $h(r,\varphi)$ is decreasing with respect to $\varphi\in\big[\,0,\,\frac{\pi}{3}\,\big]$. This means that $f(x)$ is increasing as a function of $x\in\big[\,\frac{r}{2}\,,r\big]$, i.e. the relation (26) is indeed equivalent to $a\le x$ and the proof is complete. □

Consequence 12. *Let the real numbers $a \geq b \geq c$ and $x \geq y \geq z$ be such that*

$$a+b+c = x+y+z = 0, \qquad a^2+b^2+c^2 = x^2+y^2+z^2.$$

Then, one of the following inequalities holds:

$$e^a + e^b + e^c \leq e^x + e^y + e^z \,, \tag{32}$$

or

$$e^{-a} + e^{-b} + e^{-c} \leq e^{-x} + e^{-y} + e^{-z} \,. \tag{33}$$

The inequalities (32) *and* (33) *are satisfied simultaneously if and only if $a = x$, $b = y$ and $c = z$.*

Proof. According to Lemma 11, the inequality (32) is equivalent to

$$a \leq x \,, \tag{34}$$

while the inequality (33) is equivalent to

$$-\,a \leq -x \,. \tag{35}$$

Since one of the relations (34) and (35) must hold, we have proved that one of the inequalities (32) and (33) is satisfied. They are simultaneously satisfied if and only if both (34) and (35) hold true, i.e. $a = x$ (and consequently $b = y$, $c = z$). □

Consequence 13. *Let the real numbers $a \geq b \geq c$ and $x \geq y \geq z$ be such that*

$$a+b+c = x+y+z = 0, \qquad a^2+b^2+c^2 = x^2+y^2+z^2$$
$$\text{and} \qquad e^a + e^b + e^c = e^x + e^y + e^z \,.$$

Then, we have $a = x$, $b = y$ and $c = z$.

Proof. Since by hypothesis $e^a + e^b + e^c \leq e^x + e^y + e^z$ holds, we can apply the Lemma 11 to deduce $a \leq x$ and $c \leq z$.
On the other hand, by virtue of the inverse inequality $e^x + e^y + e^z \leq e^a + e^b + e^c$ and Lemma 11 we obtain $x \leq a$ and $z \leq c$. In conclusion, we get $a = x$, $c = z$ and $b = y$. □

Proof of Theorem 10. In order to prove (13) we define the real numbers

$$t_i = k\, z_i \quad (i = 1,2,3) \qquad \text{where} \quad k = \sqrt{\frac{c_1^2+c_2^2+c_3^2}{z_1^2+z_2^2+z_3^2}} \; > 1. \tag{36}$$

Then we have

$$t_1+t_2+t_3 = c_1+c_2+c_3 = 0 \qquad \text{and} \qquad t_1^2+t_2^2+t_3^2 = c_1^2+c_2^2+c_3^2 \,. \tag{37}$$

If we apply the Consequence 12 for the numbers $c_1 \geq c_2 \geq c_3$ and $t_1 \geq t_2 \geq t_3$, then we obtain that

$$\begin{aligned} &e^{t_1} + e^{t_2} + e^{t_3} \leq e^{c_1} + e^{c_2} + e^{c_3} \qquad \text{or} \\ &e^{-t_1} + e^{-t_2} + e^{-t_3} \leq e^{-c_1} + e^{-c_2} + e^{-c_3}. \end{aligned} \tag{38}$$

In what follows, let us show that

$$e^{z_1} + e^{z_2} + e^{z_3} < e^{t_1} + e^{t_2} + e^{t_3}\,. \tag{39}$$

Using the notations $\rho := \sqrt{\frac{2}{3}\,(z_1^2 + z_2^2 + z_3^2)}$ and

$$\cos\zeta := \frac{z_1}{\rho} \in \Big[\frac{1}{2},\, 1\Big], \qquad \text{i.e.} \qquad \zeta := \arccos\Big(\frac{z_1}{\rho}\Big) \in \Big[0,\, \frac{\pi}{3}\Big],$$

we have $k\rho := \sqrt{\frac{2}{3}\,(t_1^2 + t_2^2 + t_3^2)}$ and $\cos\zeta = \dfrac{t_1}{k\rho}$. With the help of the function h defined in (27), we can write the inequality (39) in the form

$$e^{\rho\cos\zeta} + e^{\rho\cos(\zeta+2\pi/3)} + e^{\rho\cos(\zeta-2\pi/3)} < e^{k\rho\cos\zeta} + e^{k\rho\cos(\zeta+2\pi/3)} + e^{k\rho\cos(\zeta-2\pi/3)}\,, \qquad \text{or}$$

$$h(\rho,\zeta) < h(k\rho,\zeta), \qquad \forall\,(\rho,\zeta) \in (0,\infty) \times \Big[0,\, \frac{\pi}{3}\Big],\;\; k > 1. \tag{40}$$

The relation (40) asserts that the function h defined in (27) is increasing with respect to the first variable $r \in (0,\infty)$. To show this, we compute the derivative

$$\frac{\partial}{\partial r}\, h(r,\varphi) = e^{r\cos\varphi}\cos\varphi + e^{r\cos(\varphi+2\pi/3)}\cos(\varphi + \tfrac{2\pi}{3}\,) + e^{r\cos(\varphi-2\pi/3)}\cos(\varphi - \tfrac{2\pi}{3}\,). \tag{41}$$

By virtue of the Chebyshev's sum inequality we deduce from (41) that

$$\frac{\partial}{\partial r}\, h(r,\varphi)\; > 0. \tag{42}$$

Indeed, the Chebyshev's sum inequality [2, 2.17] asserts that: if $a_1 \geq a_2 \geq \ldots \geq a_n$ and $b_1 \geq b_2 \geq \ldots \geq b_n$ then

$$n\sum_{k=1}^{n} a_k b_k \;\geq\; \Big(\sum_{k=1}^{n} a_k\Big)\Big(\sum_{k=1}^{n} b_k\Big).$$

In our case, we derive the following result: for any real numbers x, y, z such that $x+y+z = 0$, the inequality

$$xe^x + ye^y + ze^z \geq \frac{1}{3}\,(x+y+z)(e^x + e^y + e^z) = 0\,, \tag{43}$$

holds true, with equality if and only if $x = y = z = 0$.

Applying the result (43) to the function (41) we deduce the relation (42). This means that $h(r,\varphi)$ is an increasing function of r, i.e. the inequality (40) holds, and hence, we have proved (39).
One can show analogously that the inequality

$$e^{-z_1} + e^{-z_2} + e^{-z_3} < e^{-t_1} + e^{-t_2} + e^{-t_3} \tag{44}$$

is also valid. From (38), (39) and (44) it follows that the assertion (13) holds true. Thus, the proof of Theorem 10 is complete. □

Since the statements of the Theorems 8 and 10 are equivalent, we have proved also the inequality (8).

Remark 14. *The inequality* (8) *becomes an equality if and only if* $z_i = c_i$, $i = 1, 2, 3$.

Proof. Indeed, assume that $z_1^2+z_2^2+z_3^2 = c_1^2+c_2^2+c_3^2$. Then, we can apply the Consequence 12 and we deduce that

$$e^{z_1} + e^{z_2} + e^{z_3} \leq e^{c_1} + e^{c_2} + e^{c_3} \quad \text{or} \quad e^{-z_1} + e^{-z_2} + e^{-z_3} \leq e^{-c_1} + e^{-c_2} + e^{-c_3}. \tag{45}$$

Taking into account $(7)_{1,2}$ in conjunction with (45) we find

$$e^{z_1} + e^{z_2} + e^{z_3} = e^{c_1} + e^{c_2} + e^{c_3} \quad \text{or} \quad e^{-z_1} + e^{-z_2} + e^{-z_3} = e^{-c_1} + e^{-c_2} + e^{-c_3}. \tag{46}$$

By virtue of (46) we can apply the Consequence 13 to derive $z_1 = c_1$ and consequently $z_2 = c_2$, $z_3 = c_3$. □

Let us prove the following version of the inequality (6) for two pairs of numbers d_1, d_2 and x_1, x_2 :

Remark 15. *If the real numbers* $d_1 \geq d_2 > 0$ *and* $x_1 \geq x_2 > 0$ *are such that*

$$x_1^2 + x_2^2 \geq d_1^2 + d_2^2 \qquad \textit{and} \qquad x_1 x_2 \ = \ d_1 d_2 \ = 1\,, \tag{47}$$

then the inequality

$$(\log x_1)^2 + (\log x_2)^2 \geq (\log d_1)^2 + (\log d_2)^2 \tag{48}$$

holds true. Note that the additional condition

$$\frac{1}{x_1^2} + \frac{1}{x_2^2} \geq \frac{1}{d_1^2} + \frac{1}{d_2^2}$$

is automatically fulfilled.

Proof. Since $x_1x_2 = d_1d_2 = 1$ and $d_1 \ge d_2 > 0$, $x_1 \ge x_2 > 0$, we have $x_1 \ge 1$, $d_1 \ge 1$ and

$$\log x_1 = -\log x_2 \ge 0\,, \qquad \log d_1 = -\log d_2 \ge 0\,,$$

so that the inequality (48) is equivalent to $\log x_1 \ge \log d_1$, i.e. we have to show that $x_1 \ge d_1$.

Indeed, if we insert $x_2 = \dfrac{1}{x_1}$ and $d_2 = \dfrac{1}{d_1}$ into the inequality $(47)_1$ then we find

$$x_1^2 + \frac{1}{x_1^2} \ \ge\ d_1^2 + \frac{1}{d_1^2}\ ,$$

which means that $x_1 \ge d_1$ since the function $t \mapsto t^2 + \dfrac{1}{t^2}$ is increasing for $t \in [1, \infty)$. This completes the proof. $\square$

Alternative proof of Remark 15. Let $x_3 = d_3 = 1$. Then (47) implies $x_1^2 + x_2^2 + x_3^2 \ge d_1^2 + d_2^2 + d_3^2$ and $x_1x_2x_3 = d_1d_2d_3 = 1$ as well as

$$x_1^2x_2^2 + x_2^2x_3^2 + x_1^2x_3^2 = 1 + x_2^2 + x_1^2 \ge 1 + d_2^2 + d_1^2 = d_1^2d_2^2 + d_2^2d_3^2 + d_1^2d_3^2, \tag{49}$$

because $x_1^2x_2^2 = 1 = d_1^2d_2^2$, and Theorem 6 provides the assertion. $\square$

4 Some counterexamples for weakened assumptions

Example 16. Unlike in the 2D case in Remark 15, for two triples of numbers the second condition $(18)_2$ of Theorem 2, namely $y_1y_2 + y_2y_3 + y_1y_3 \ge a_1a_2 + a_2a_3 + a_1a_3$, cannot be removed. Let

$$y_1 = e^6, y_2 = 1, y_3 = e^{-6}, a_1 = e^4, a_2 = e^4, a_3 = e^{-8}.$$

Then $y_1y_2y_3 = a_1a_2a_3 = 1$ and

$$y_1 + y_2 + y_3 > e^6 > e^2e^4 > 3e^4 > a_1 + a_2 + a_3,$$

but

$$(\log y_1)^2 + (\log y_2)^2 + (\log y_3)^2 = 36 + 0 + 36 < 16 + 16 + 64 = (\log a_1)^2 + (\log a_2)^2 + (\log a_3)^2.$$

Example 17. The condition $y_1y_2y_3 = a_1a_2a_3$ cannot be weakened to $y_1y_2y_3 \ge a_1a_2a_3$. Indeed, let $y_2 = y_3 = a_1 = a_2 = 1$, $y_1 = e$, $a_3 = e^{-2}$. Then

$$\begin{aligned} y_1 + y_2 + y_3 = e + 1 + 1 &\ge 1 + 1 + e^{-2} = a_1 + a_2 + a_3\,, \\ y_1y_2 + y_1y_3 + y_2y_3 = e + e + 1 &\ge 1 + e^{-2} + e^{-2} = a_1a_2 + a_1a_3 + a_2a_3\,, \\ y_1y_2y_3 = e &\ge e^{-2} = a_1a_2a_3\,. \end{aligned}$$

But nevertheless

$(\log y_1)^2 + (\log y_2)^2 + (\log y_3)^2 = 1 + 0 + 0 < 0 + 0 + 4 = (\log a_1)^2 + (\log a_2)^2 + (\log a_3)^2.$

A counterexample for the two variable case can be constructed analogously.

Example 18. Even with an analogous condition, the inequality (4) does not hold for $n = 4$ numbers (without further assumptions). Indeed, let

$$y_1 = e, y_2 = y_3 = e^7, y_4 = e^{-15}, a_1 = a_2 = e^6, a_3 = e^7, a_4 = e^{-19}.$$

Then $y_1y_2y_3y_4 = a_1a_2a_3a_4 = 1$. Also

$$y_1 + y_2 + y_3 + y_4 = e + e^7 + e^7 + e^{-15} > 0 + e^7 + 2e^6 + e^{-19} = a_1 + a_2 + a_3 + a_4.$$

Furthermore

$$y_1y_2 + y_1y_3 + y_1y_4 + y_2y_3 + y_2y_4 + y_3y_4 = e^8 + e^8 + e^{-14} + e^{14} + e^{-8} + e^{-8}$$

and

$$a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4 = e^{12} + e^{13} + e^{-13} + e^{13} + e^{-13} + e^{-12}.$$

Since $e^2 > 2e + 1$, we have $e^{14} > e^{13} + e^{13} + e^{12}$ and therefore

$$y_1y_2 + y_1y_3 + y_1y_4 + y_2y_3 + y_2y_4 + y_3y_4 \geq a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4.$$

Nevertheless, for the sum of squared logarithms, the "reverse" inequality

$$\begin{aligned}(\log y_1)^2 + (\log y_2)^2 + (\log y_3)^2 + (\log y_4)^2 = 1 + 49 + 49 + 225 = 324 \\ < 482 = 36 + 36 + 49 + 361 = (\log a_1)^2 + (\log a_2)^2 + (\log a_3)^2 + (\log a_4)^2\end{aligned}$$

holds true.

Example 19. The inequality (4) does not remain true either, if the function $\log(y)$ is replaced by its linearization $(y - 1)$. Indeed, let $y_1 = 9$, $y_2 = 5$, $y_3 = \frac{1}{45}$, $a_1 = 10$, $a_2 = 1$, $a_3 = \frac{1}{10}$. Then

$$y_1 + y_2 + y_3 > 14 > 11.1 = a_1 + a_2 + a_3$$

and

$$y_1y_2 + y_1y_3 + y_2y_3 > 45 \geq 11.1 = a_1a_2 + a_1a_3 + a_2a_3.$$

But

$$\begin{aligned}(y_1 - 1)^2 + (y_2 - 1)^2 + (y_3 - 1)^2 = 64 + 16 + \left(\frac{44}{45}\right)^2 \\ < 81 < 9^2 + 0 + \left(\frac{9}{10}\right)^2 = (a_1 - 1)^2 + (a_2 - 1)^2 + (a_3 - 1)^2.\end{aligned}$$

5 Conjecture for arbitrary n

The structure of the inequality in dimensions $n = 2$ and $n = 3$ and extensive numerical sampling strongly suggest that the inequality holds for all $n \in \mathbb{N}$ if the n corresponding conditions are satisfied, more precisely, in terms of the elementary symmetric polynomials

Conjecture 20. *Let $n \in \mathbb{N}$ and $y_i, a_i > 0$ for $i = 1, \ldots, n$. If for all $i = 1, \ldots, n-1$ we have*

$$e_i(y_1, \ldots, y_n) \geq e_i(a_1, \ldots, a_n) \text{ and } e_n(y_1, \ldots, y_n) = e_n(a_1, \ldots, a_n),$$

then

$$\sum_{i=1}^{n} (\log y_i)^2 \geq \sum_{i=1}^{n} (\log a_i)^2.$$

6 Applications

The investigation in this paper has been motivated by some recent applications. The new sum of squared logarithm inequality is one of the fundamental tools in deducing a novel optimality result in matrix analysis and the conditions in the form (3) had been deduced in the course of that work. Optimality in the matrix problem suggested the sum of squared logarithm inequality. Indeed, based on the present result in [6] it has been shown that for all invertible $Z \in \mathbb{C}^{3\times 3}$ and for any definition of the matrix logarithm as possibly multivalued solution $X \in \mathbb{C}^{3\times 3}$ of $\exp X = Z$ it holds

$$\begin{aligned} \min_{Q^*Q=I} \|\log Q^* Z\|_F^2 &= \left\|\log U_p^* Z\right\|_F^2 = \|\log H\|_F^2 \,, \\ \min_{Q*Q=I} \|\operatorname{sym} \log Q^* Z\|_F^2 &= \left\|\operatorname{sym} \log U_p^* Z\right\|_F^2 = \|\log H\|_F^2 \,, \end{aligned} \tag{50}$$

where $\operatorname{sym} X = \frac{1}{2}(X + X^*)$ is the Hermitian part of $X \in \mathbb{C}^{3\times 3}$ and U_p is the unitary factor in the polar decomposition of Z into unitary and Hermitian positive definite matrix H

$$Z = U_p \, H \,. \tag{51}$$

This result (50) generalizes the fact that for any complex logarithm and for all $z \in \mathbb{C} \setminus \{0\}$

$$\min_{\vartheta \in (-\pi,\pi]} |\log_{\mathbb{C}}[e^{-i\vartheta} z]|^2 = |\log_{\mathbb{R}} |z||^2 \,, \qquad \min_{\vartheta \in (-\pi,\pi]} |\mathfrak{Re} \log_{\mathbb{C}}[e^{-i\vartheta} z]|^2 = |\log_{\mathbb{R}} |z||^2 \,. \tag{52}$$

The optimality result (50) can now also be viewed as another characterization of the unitary factor in the polar decomposition. In addition, in a forthcoming contribution [7] we use (50) to calculate the geodesic distance of the isochoric part of the deformation

gradient $\frac{F}{\det F^{\frac{1}{3}}} \in \mathrm{SL}(3,\mathbb{R}))$ to $\mathrm{SO}(3,\mathbb{R}))$ in the canonical left-invariant Riemannian metric on $\mathrm{SL}(3,\mathbb{R})$, to the effect that

$$\mathrm{dist}^2_{\mathrm{geod}}\left(\frac{F}{\det F^{\frac{1}{3}}}, \mathrm{SO}(3,\mathbb{R})\right) = \left\| \mathrm{dev}_3 \log \sqrt{F^T F} \right\|^2_F . \tag{53}$$

where $\mathrm{dev}_3 X = X - \frac{1}{3} \operatorname{tr} X \, I$ is the orthogonal projection of $X \in \mathbb{R}^{3\times 3}$ to trace free matrices. Thereby, we provide a rigorous geometric justification for the preferred use of the Hencky-strain measure $\left\| \log \sqrt{F^T F} \right\|^2_F$ in nonlinear elasticity and plasticity theory [3].

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